

## Two-dimensional extremal black holes

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### Abstract

We discuss the two-dimensional (2D),  $\epsilon < 2$  extremal ground states of charged black hole. Here  $\epsilon$  is the dilaton coupling parameter for the Maxwell term. The complete analysis of stability is carried out for all these extremal black holes. It is found that they are all unstable. To understand this instability, we study the non-extremal charged black hole with two (inner and outer) horizons. The extremal black holes appear when two horizons coalesce. It conjectures that the instability originates from the inner horizon.

## I. INTRODUCTION

Recently the extremal black holes have received much attention. Extremal black holes provide a simple laboratory in which to investigate the quantum aspects of black hole [1]. One of the crucial features is that the Hawking temperature vanishes. The black hole with  $M > Q$  will radiate down to its extremal  $M = Q$  state. Thus the extremal black hole may play a role of the stable endpoint for the Hawking evaporation. It has been also proposed that although the extremal black hole has nonzero area, it has zero entropy [2]. This is because the extremal case is distinct topologically from the nonextremal one.

It is very important to inquire into the stability of the extremal black holes, which is essential to establish their physical existence. We note that the 4D extremal charged black holes with the coupling parameter ( $a$ ) are shown to be classically stable [3]. The  $a = 0$  case corresponds to the extremal Reissner-Nordström black hole. Since all potentials are positive definite outside the horizon, one can easily infer the stability of 4D extremal charged black holes using the same argument as employed by Chandrasekhar [4]. To the contrary, it appears that the 2D extremal black holes are unstable [5].

In this paper, we consider the prototype model for extremal black holes. This is the 2D,  $\epsilon < 2$  ( $\epsilon$  is the 2D coupling parameter corresponding to  $a$ ) extremal black holes [6]. It is very important to find whether these black holes are stable or not. For this end, one reminds that an extremal black hole is regarded as the limit of a non-extremal charged black hole. The 2D charged black hole has the event (outer) as well as the Cauchy (inner) horizons. The stability of the outer horizon is followed by the conventional argument of the stability. One easy way of understanding a black hole is to find out how it reacts to external perturbations. We always visualize the black hole as presenting an effective potential barrier (or well) to the on-coming waves [4]. As a compact criterion for a single horizon (for an extremal black hole) or outer horizon, the horizon is unstable if there exists the potential well to the on-coming waves [7]. This is so because the Schrödinger-type equation with the potential well always allows the bound state as well as scattering states. The former shows up as an imaginary

frequency mode, leading to an exponentially growing mode. If one finds any exponentially growing mode, the extremal black hole (or the outer horizon of non-extremal black hole) is unstable. In Ref.[8] a conformally coupled scalar ( $f_i$ ) is used as a test field, while in our case a tachyon is used as a test field. If one takes a conformally coupled scalar to study the black hole, one finds the free field equation for the linearization :  $\bar{\nabla}^2 f_i = 0 \rightarrow (d^2/dr^{*2} + \omega^2)f_i = 0$ . This implies that one cannot find the potentials, which are crucial for obtaining information about the black holes. Although  $f_i$  is useful for a semiclassical study of the black hole, it is not a good field for our study.

On the other hand, the stability analysis of the inner horizon is related to how waves from the external world (entering the inner region between two horizons via the outer one) develop as they approach the inner one and affect a free-falling observer (FFO). If a FFO crossing the inner horizon meets an infinitely infalling radiation, one concludes that the inner horizon is unstable. This is a direct consequence of the non-trivial causal structure of the space-time: An observer crossing the inner horizon sees the entire history of the universe in a tremendous flash. We will show that the inner horizons of the 2D,  $\epsilon < 2$  charged black holes are unstable, whereas the outer horizons are stable. One finds the barrier-well type potentials ( $V^{IN}$ ) between the Cauchy horizon and event horizon, while a potential barrier ( $V^{OUT}$ ) is induced outside the event horizon. When these coalesce (extremal case), a barrier-well type potential persists outside the event horizon. This induces the instability of the extremal black hole.

The organization of this paper is as follows. We discuss the 2D,  $\epsilon < 2$  black hole solutions in Sec.II. Analyzing the metric function  $f(r, \epsilon, Q)$ , one finds the extremal, and non-extremal black holes. For the stability analysis, in Sec.III we linearize the equation of motion around the background solution. Sec. IV is devoted to investigate the stability for extremal black holes. It turns out that they are all unstable. In order to understand this instability, we study the non-extremal black holes in Sec. V. The inner structure of charged black hole is explored to understand the nature of the Cauchy horizon. The Hawking temperature is calculated in Sec. VI. Finally we discuss our results in Sec. VII.

## II. THE $\epsilon < 2$ BLACK HOLE SOLUTION

We start with two-dimensional dilaton gravity  $(\Phi, G_{\mu\nu})$  conformally coupled to Maxwell  $(F_{\mu\nu})$  and tachyon  $(T)$ [10]

$$S_{l-e} = \int d^2x \sqrt{-G} e^{-2\Phi} \{ R + 4(\nabla\Phi)^2 + \alpha^2 - \frac{1}{2}e^{\epsilon\Phi} F^2 - \frac{1}{2}(\nabla T)^2 + T^2 \}. \quad (1)$$

The above action with  $\epsilon = 0$  can be realized from the heterotic string. We introduce the tachyon to study the properties of black holes in a simple way. Setting  $\alpha^2 = 8$  and after deriving equations of motion, we take the transformation

$$-2\Phi \rightarrow \Phi, \quad T \rightarrow \sqrt{2}T, \quad -R \rightarrow R. \quad (2)$$

Then the equations of motion become

$$R_{\mu\nu} + \nabla_\mu \nabla_\nu \Phi + \nabla_\mu T \nabla_\nu T + \frac{4-\epsilon}{4} e^{-\epsilon\Phi/2} F_{\mu\rho} F_\nu{}^\rho = 0, \quad (3)$$

$$\nabla^2 \Phi + (\nabla\Phi)^2 - \frac{1}{2} e^{-\epsilon\Phi/2} F^2 - 2T^2 - 8 = 0, \quad (4)$$

$$\nabla_\mu F^{\mu\nu} + \frac{2-\epsilon}{2} (\nabla_\mu \Phi) F^{\mu\nu} = 0, \quad (5)$$

$$\nabla^2 T + \nabla_\mu \Phi \nabla^\mu T + 2T = 0. \quad (6)$$

An electrically charged black hole solution to the above equations is obtained as

$$\bar{\Phi} = 2\sqrt{2}r, \quad \bar{F}_{tr} = Q e^{-(2-\epsilon)\sqrt{2}r}, \quad \bar{T} = 0, \quad \bar{G}_{\mu\nu} = \begin{pmatrix} -f & 0 \\ 0 & f^{-1} \end{pmatrix}, \quad (7)$$

with the metric function

$$f = 1 - \frac{M}{\sqrt{2}} e^{-2\sqrt{2}r} + \frac{Q^2}{4(2-\epsilon)} e^{-(4-\epsilon)\sqrt{2}r}. \quad (8)$$

Here  $M$  and  $Q$  are the mass and electric charge of the black hole, respectively. We hereafter take  $M = \sqrt{2}$  for convenience. Note that from the requirement of the finiteness of electric energy ( $\bar{F}(r \rightarrow \infty) \rightarrow 0$ ) and the asymptotically flat spacetime ( $f(r \rightarrow \infty) \rightarrow 1$ ), we have the important constraint :  $\epsilon < 2$ .

We analyze the metric function  $f(r, \epsilon, Q)$  in (8) explicitly. In general, from  $f = 0$  we can obtain two roots ( $r_{\pm}$ ) where  $r_+$  ( $r_-$ ) correspond to the event (Cauchy) horizons of the charged black hole. One is interested in the extremal limit (multiple root:  $r_+ = r_- \equiv r_o$ ) of the charged holes. This may provide a toy model to investigate the late stages of Hawking evaporation. First of all, we wish to obtain the condition for  $f(r, \epsilon, Q) = 0$  to have the multiple root. For  $\epsilon < 2$ , the shape of  $f(r, \epsilon, Q)$  is always concave ( $\smile$ ). Thus the multiple root is always obtained when  $f = 0$  and  $f' = 0$ , which imply that  $Q^2(\epsilon) = Q_e^2(\epsilon) = 8(\frac{2-\epsilon}{4-\epsilon})^{(4-\epsilon)/2}$ . Here the prime ( $\prime$ ) denotes the derivative with respect to  $r$ . The extremal horizon is located at

$$r_o(\epsilon) = -\frac{1}{2\sqrt{2}} \log\left(\frac{4-\epsilon}{2-\epsilon}\right). \quad (9)$$

The explicit form of the extremal  $f_e$  is

$$f_e(r, \epsilon) = 1 - e^{-2\sqrt{2}r} + \frac{2}{(2-\epsilon)} \left(\frac{2-\epsilon}{4-\epsilon}\right)^{(4-\epsilon)/2} e^{-(4-\epsilon)\sqrt{2}r}. \quad (10)$$

Fig. 1 shows the shapes of  $f(r, \epsilon, Q)$  for three cases :  $Q^2 < Q_e^2$ ,  $Q^2 = Q_e^2$  and  $Q^2 > Q_e^2$  for  $\epsilon = 0.5$ . For  $Q^2 < Q_e^2$ , we find the non-extremal black hole. One always obtains two roots( $r_{\pm}$ ) :  $f(r_+, \epsilon, Q) = f(r_-, \epsilon, Q) = 0$ . The analytic forms of  $r_{\pm}$  are not known because of the complex form of the metric function. However the numerical values of  $r_{\pm}$  can be obtained for given  $\epsilon$  and  $Q$ . Hence our information for the metric function  $f(r, \epsilon, Q)$  comes mainly from the graphical analysis. In the case of  $Q^2 > Q_e^2$ , there does not exist any solution to  $f(r, \epsilon, Q) = 0$ , which means that there is no horizon. Hereafter we no longer consider this case.

### III. LINEAR PERTURBATION

We introduce small perturbation fields around the background solution as [5]

$$F_{tr} = \bar{F}_{tr} + \mathcal{F}_{tr} = \bar{F}_{tr} \left[1 - \frac{\mathcal{F}(r, t)}{Q}\right], \quad (11)$$

$$\Phi = \bar{\Phi} + \phi(r, t), \quad (12)$$

$$G_{\mu\nu} = \bar{G}_{\mu\nu} + h_{\mu\nu} = \bar{G}_{\mu\nu}[1 - h(r, t)], \quad (13)$$

$$T = \bar{T} + \hat{t} \equiv \exp(-\frac{\bar{\Phi}}{2})[0 + t(r, t)]. \quad (14)$$

One has to linearize (3)-(6) in order to obtain the equations governing the perturbations as

$$\begin{aligned} \delta R_{\mu\nu}(h) + \bar{\nabla}_\mu \bar{\nabla}_\nu \phi - \delta \Gamma_{\mu\nu}^\rho(h) \bar{\nabla}_\rho \bar{\Phi} + \frac{4-\epsilon}{2} e^{-\epsilon\sqrt{2}r} \bar{F}_{\mu\rho} \mathcal{F}_\nu{}^\rho - \frac{4-\epsilon}{4} e^{-\epsilon\sqrt{2}r} \bar{F}_{\mu\rho} \bar{F}_{\nu\alpha} h^{\rho\alpha} \\ - \frac{\epsilon(4-\epsilon)}{8} e^{-\epsilon\sqrt{2}r} \bar{F}_{\mu\rho} \bar{F}_\nu{}^\rho \phi = 0, \end{aligned} \quad (15)$$

$$\begin{aligned} \bar{\nabla}^2 \phi - h^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\Phi} - \bar{G}^{\mu\nu} \delta \Gamma_{\mu\nu}^\rho(h) \partial_\rho \bar{\Phi} - h^{\mu\nu} \partial_\mu \bar{\Phi} \partial_\nu \bar{\Phi} + 2\bar{G}^{\mu\nu} \partial_\mu \bar{\Phi} \partial_\nu \phi - e^{-\epsilon\sqrt{2}r} \bar{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \\ + e^{-\epsilon\sqrt{2}r} \bar{F}_{\mu\nu} \bar{F}_\rho{}^\nu h^{\mu\rho} + \frac{\epsilon}{4} e^{-\epsilon\sqrt{2}r} \bar{F}^2 \phi = 0, \end{aligned} \quad (16)$$

$$(\bar{\nabla}_\mu + \frac{2-\epsilon}{2} \partial_\mu \bar{\Phi})(\mathcal{F}^{\mu\nu} - \bar{F}_\alpha{}^\nu h^{\alpha\mu} - \bar{F}^\mu{}_\beta h^{\beta\nu}) + \bar{F}^{\mu\nu}(\delta \Gamma_{\sigma\mu}^\sigma(h) + \frac{2-\epsilon}{2}(\partial_\mu \phi)) = 0, \quad (17)$$

$$\bar{\nabla}^2 \hat{t} + \bar{\nabla}_\mu \bar{\Phi} \bar{\nabla}^\mu \hat{t} + 2\hat{t} = 0, \quad (18)$$

where

$$\delta R_{\mu\nu}(h) = \frac{1}{2} \bar{\nabla}_\mu \bar{\nabla}_\nu h^\rho{}_\rho + \frac{1}{2} \bar{\nabla}^\rho \bar{\nabla}_\rho h_{\mu\nu} - \frac{1}{2} \bar{\nabla}^\rho \bar{\nabla}_\nu h_{\rho\mu} - \frac{1}{2} \bar{\nabla}^\rho \bar{\nabla}_\mu h_{\nu\rho}, \quad (19)$$

$$\delta \Gamma_{\mu\nu}^\rho(h) = \frac{1}{2} \bar{G}^{\rho\sigma} (\bar{\nabla}_\nu h_{\mu\sigma} + \bar{\nabla}_\mu h_{\nu\sigma} - \bar{\nabla}_\sigma h_{\mu\nu}). \quad (20)$$

From (17) one can express  $\mathcal{F}$  in terms of  $\phi$  and  $h$  as

$$\mathcal{F} = -Q(h + \frac{2-\epsilon}{2}\phi). \quad (21)$$

This means that  $\mathcal{F}$  is no longer an independent mode. Also from the diagonal element of (15), we have

$$\bar{\nabla}^2 h - 2\bar{\nabla}_t^2 \phi - 2\sqrt{2}\partial^r h - (4-\epsilon)e^{-\epsilon\sqrt{2}r} \bar{F}_{tr} \mathcal{F}^{tr} - \frac{(4-\epsilon)}{2} e^{-\epsilon\sqrt{2}r} \bar{F}_{tr} \bar{F}^{tr} (h - \frac{\epsilon}{2}\phi) = 0, \quad (22)$$

$$\bar{\nabla}^2 h - 2\bar{\nabla}_r^2 \phi + 2\sqrt{2}\partial^r h - (4-\epsilon)e^{-\epsilon\sqrt{2}r} \bar{F}_{tr} \mathcal{F}^{tr} - \frac{(4-\epsilon)}{2} e^{-\epsilon\sqrt{2}r} \bar{F}_{tr} \bar{F}^{tr} (h - \frac{\epsilon}{2}\phi) = 0. \quad (23)$$

Adding the above two equations leads to

$$\bar{\nabla}^2 (h - \phi) - \frac{Q^2(4-\epsilon)}{2} e^{-(4-\epsilon)\sqrt{2}r} (h + \frac{4-\epsilon}{2}\phi) = 0. \quad (24)$$

Also the dilaton equation (16) leads to

$$\bar{\nabla}^2 \phi + 4\sqrt{2}f\phi' + 2\sqrt{2}(f' + 2\sqrt{2}f)h - \frac{Q^2(4-\epsilon)}{2}e^{-(4-\epsilon)\sqrt{2}r}\phi = 0. \quad (25)$$

And the off-diagonal element of (15) takes the form

$$\partial_t\{(\partial_r - \Gamma_{tr}^t)\phi + \sqrt{2}h\} = 0, \quad (26)$$

which provides us the relation between  $\phi$  and  $h$  as

$$\phi' = -\sqrt{2}h + \frac{1}{2}\frac{f'}{f}\phi + U(r). \quad (27)$$

Here  $U(r)$  is the residual gauge degrees of freedom and thus we set  $U(r) = 0$  for simplicity.

Substituting (27) into (25), we have

$$\bar{\nabla}^2 \phi + 2\sqrt{2}f'(h + \phi) - \frac{Q^2(4-\epsilon)}{2}e^{-(4-\epsilon)\sqrt{2}r}\phi = 0. \quad (28)$$

Calculating (24) + 2 × (28), one finds the other equation

$$\bar{\nabla}^2(h + \phi) + 4\sqrt{2}f'(h + \phi) - 2Q^2e^{-(4-\epsilon)\sqrt{2}r}\left\{(h + 4\phi) - \frac{\epsilon}{4}\left(h + \frac{12-\epsilon}{2}\phi\right)\right\} = 0. \quad (29)$$

Although (24) and (29) look like the very complicated forms, these reduce to

$$\bar{\nabla}^2(h - \phi) = 0, \quad (30)$$

$$\bar{\nabla}^2(h + \phi) + 4\sqrt{2}f'(h + \phi) = 0 \quad (31)$$

in the asymptotically flat region ( $r \rightarrow \infty$ ). This suggests that one obtains two graviton-dilaton modes. However, it is important to check whether the graviton ( $h$ ), dilaton ( $\phi$ ), Maxwell mode ( $\mathcal{F}$ ) and tachyon ( $t$ ) are physically propagating modes in the 2D charged black hole background. We review the conventional counting of degrees of freedom. The number of degrees of freedom for the gravitational field ( $h_{\mu\nu}$ ) in  $D$ -dimensions is  $(1/2)D(D-3)$ . For a Schwarzschild black hole, we obtain two degrees of freedom. These correspond to the Regge-Wheeler mode for odd-parity perturbation and Zerilli mode for even-parity perturbation [4]. We have  $-1$  for  $D = 2$ . This means that in two dimensions the contribution of the graviton

is equal and opposite to that of a spinless particle (dilaton). The graviton-dilaton modes  $(h + \phi, h - \phi)$  are gauge degrees of freedom and thus turn out to be nonpropagating modes [6]. In addition, the Maxwell field has  $D - 2$  physical degrees of freedom. The Maxwell field has no physical degrees of freedom for  $D = 2$ . Actually from (21) it turns out to be a redundant one. Since all these are nonpropagating modes, it is necessary to consider the remaining one (18). The tachyon as a spectator is a physically propagating mode. This is used to illustrate many of the qualitative results about the 2D charged black hole in a simpler context. Its linearized equation is

$$f^2 \partial_r^2 t + f \partial_r f \partial_r t - \{\sqrt{2} f \partial_r f - 2f(1 - f)\} t - \partial_t^2 t = 0. \quad (32)$$

To study the physical implications, the above equation should be transformed into one-dimensional Schrödinger-type equation. Introducing a tortoise coordinate

$$r \rightarrow r^* \equiv g(r),$$

(32) can be rewritten as

$$f^2 g'^2 \frac{\partial^2}{\partial r^{*2}} t + f(f g'' + f' g') \frac{\partial}{\partial r^*} t - \{\sqrt{2} f f' - 2f(1 - f)\} t - \frac{\partial^2}{\partial t^2} t = 0. \quad (33)$$

Requiring that the coefficient of the linear derivative vanish, one finds the relation

$$g' = \frac{1}{f}. \quad (34)$$

Assuming  $t_\omega(r^*, t) \sim \tilde{t}_\omega(r^*) e^{i\omega t}$ , one can cast (33) into the Schrödinger-type equation

$$\left\{ \frac{d^2}{dr^{*2}} + \omega^2 - V(r, \epsilon, Q) \right\} \tilde{t}_\omega = 0, \quad (35)$$

where the effective potential  $V(r, \epsilon, Q)$  is given by

$$V(r, \epsilon, Q) = f(\sqrt{2} f' - 2(1 - f)). \quad (36)$$



#### IV. THE EXTREMAL CASE

First we consider the extremal black holes. The potentials surrounding the extremal black holes are given by

$$V^e(r, \epsilon) = 2e^{-2\sqrt{2}r} f_e(r, \epsilon) \left\{ 1 - 2\left(\frac{3-\epsilon}{2-\epsilon}\right)\left(\frac{2-\epsilon}{4-\epsilon}\right)^{(4-\epsilon)/2} e^{-(2-\epsilon)\sqrt{2}r} \right\}. \quad (37)$$

After a concrete analysis, one finds the barrier-well type potentials for  $\epsilon < 2$ . For examples, Fig. 2 shows the shapes of potentials for  $\epsilon = 1.9, 0.5$ , and  $-3$ . In this case the roots of  $V^e = 0$  are  $r = r_o, r_b$ , and  $\infty$  in sequence. Here the extremal horizon ( $r = r_o$ ) comes from  $f_e = 0$  and  $r = r_b, \infty$  from  $\sqrt{2}f'_e - 2(1 - f_e) = 0$ . Now let us translate the potential  $V^e(r, \epsilon)$  into  $V^e(r^*, \epsilon)$ . From (10) and (34), one can find the form of  $r^* = g = \int^r dr/f_e$ . Setting  $y = e^{-2\sqrt{2}r}$ , we integrate this as

$$r^* = r + \frac{1}{2\sqrt{2}(4-\epsilon)} \log |f_e| - \frac{2-\epsilon}{2\sqrt{2}(4-\epsilon)} \int^y \frac{dy}{1-y+Ay^{1+B}} \quad (38)$$

with  $A = \frac{2}{(2-\epsilon)}\left(\frac{2-\epsilon}{4-\epsilon}\right)^{(4-\epsilon)/2}$  and  $B = 1 - \frac{\epsilon}{2}$ . Since both the forms of  $V^e(r, \epsilon)$  and  $r^*$  are very complicated, we are far from obtaining the exact form of  $V^e(r^*, \epsilon)$ . But one can obtain the approximate forms to  $V^e(r^*, \epsilon)$  near the both ends. Since the second and last terms in (38) approach zero as  $r \rightarrow \infty$ , one finds that

$$r^* \simeq r. \quad (39)$$

Then (37) takes the asymptotic form

$$V_{r^* \rightarrow \infty}^e(r^*) \simeq 2 \exp(-2\sqrt{2}r^*), \quad (40)$$

which is independent of  $\epsilon$ . On the other hand, near the horizon ( $r = r_o$ ) the last term in (38) completely dominates over the first two terms. Expanding  $1 - y + Ay^{1+B}$  in a Taylor series about  $y_0 = \exp(-2\sqrt{2}r_o)$  leads to  $\frac{AB(1-B)}{2}y_o^{B-1}(y - y_o)^2$ . Plugging this into (38), one has

$$r^* \simeq -\frac{2}{\sqrt{2}(2-\epsilon)} \frac{1}{\left(\frac{4-\epsilon}{2-\epsilon} - e^{-2\sqrt{2}r}\right)}. \quad (41)$$

From (41), one finds the asymptotic relation

$$e^{-2\sqrt{2}r} \simeq \frac{4-\epsilon}{2-\epsilon} + \frac{2}{\sqrt{2}(2-\epsilon)} \frac{1}{r^*}. \quad (42)$$

Substituting this into (37) leads to the potential near the horizon ( $r \rightarrow r_o, r^* \rightarrow -\infty$ )

$$V_{r^* \rightarrow -\infty}^e(r^*, \epsilon) \simeq -\frac{1}{(4-\epsilon)} \frac{1}{r^{*2}}. \quad (43)$$

Using (40) and (43) one can construct the approximate form  $V_{app}^e(r^*, \epsilon)$  (Fig. 3). This is also a barrier-well which is localized at the origin of  $r^*$ . Our analysis is based on the approximate equation,

$$\left\{ \frac{d^2}{dr^{*2}} + \omega^2 - V_{app}^e(r^*, \epsilon) \right\} \tilde{t}_\omega^e = 0. \quad (44)$$

As is well known, two kinds of solutions to Schrödinger-type equation with potential well correspond to the bound and scattering states. In our case  $V_{app}^e(r^*, \epsilon)$  admits two solutions depending on the sign of  $\omega^2$ : (i) For  $\omega^2 > 0$  ( $\omega = \text{real}$ ), the asymptotic solution for  $\tilde{t}_\omega^e$  is given by  $\tilde{t}_{\omega, \infty}^e = \exp(i\omega r^*) + R \exp(-i\omega r^*), r^* \rightarrow \infty$  and  $\tilde{t}_{\omega, EH}^e = T \exp(i\omega r^*), r^* \rightarrow -\infty$ . Here  $R$  and  $T$  are the scattering amplitudes of two waves which are reflected and transmitted by the potential  $V_{app}^e(r^*, \epsilon)$ , when a tachyonic wave of unit amplitude with the frequency  $\omega$  is incident on the black hole from infinity. (ii) For  $\omega^2 < 0$  ( $\omega = -i\alpha$ ,  $\alpha$  is positive and real), we have the bound state. Eq. (44) is given by

$$\frac{d^2}{dr^{*2}} \tilde{t}^e = (\alpha^2 + V_{app}(r^*)) \tilde{t}^e. \quad (45)$$

The asymptotic solution is  $\tilde{t}_\infty^e \sim \exp(\pm \alpha r^*), r^* \rightarrow \infty$  and  $\tilde{t}_{EH}^e \sim \exp(\pm \alpha r^*), r^* \rightarrow -\infty$ . To ensure that the perturbation falls off to zero for large  $r^*$ , we choose  $\tilde{t}_\infty^e \sim \exp(-\alpha r^*)$ . In the case of  $\tilde{t}_{EH}^e$ , the solution  $\exp(\alpha r^*)$  goes to zero as  $r^* \rightarrow -\infty$ . Now let us observe whether or not  $\tilde{t}_{EH}^e \sim \exp(\alpha r^*)$  can be matched to  $\tilde{t}_\infty^e \sim \exp(-\alpha r^*)$ . Assuming  $\tilde{t}^e$  to be positive, the sign of  $d^2 \tilde{t}^e / dr^{*2}$  can be changed from  $+$  to  $-$  as  $r^*$  goes from  $\infty$  to  $-\infty$ . If we are to connect  $\tilde{t}_{EH}^e$  at one end to a decreasing solution  $\tilde{t}_\infty^e$  at the other, there must be a point ( $d^2 \tilde{t}^e / dr^{*2} < 0, d\tilde{t}^e / dr^* = 0$ ) at which the signs of  $\tilde{t}^e$  and  $d^2 \tilde{t}^e / dr^{*2}$  are opposite :

this is compatible with the shape of  $V_{app}^e(r^*, \epsilon)$  in Fig. 3. Thus it is possible for  $\tilde{t}_{EH}^e$  to be connected to  $\tilde{t}_\infty^e$  smoothly. Therefore a bound state solution is given by

$$\tilde{t}_\infty^e \sim \exp(-\alpha r^*), \quad (r^* \rightarrow \infty) \quad (46)$$

$$\tilde{t}_{EH}^e \sim \exp(\alpha r^*), \quad (r^* \rightarrow -\infty). \quad (47)$$

This is a regular solution everywhere in space at the initial time  $t = 0$ . It is well-known that in quantum mechanics, the bound state solution is always allowed if there exists a potential well. The time evolution of the solution with  $\omega = -i\alpha$  implies  $t_\infty^e(r^*, t) = \tilde{t}_\infty^e(r^*) \exp(-i\omega t) \sim \exp(-\alpha r^*) \exp(\alpha t)$  and  $t_{EH}^e(r^*, t) = \tilde{t}_{EH}^e(r^*) \exp(-i\omega t) \sim \exp(\alpha r^*) \exp(\alpha t)$ . This means that there exists an exponentially growing mode with time. Therefore, the  $\epsilon < 2$  extremal ground states are classically unstable.

## V. THE NON-EXTREMAL CASE

The origin of this instability comes from the barrier-well potentials. These potentials appear in all  $\epsilon < 2$  extremal black holes. This potential persists when the non-extremal black hole approaches the extremal limit (see Fig. 4). An extremal black hole is considered as the limit of a non-extremal one. A non-extremal black hole has an outer (event) and an inner (Cauchy) horizon, and these come together in the extremal limit. In this sense, it is necessary to investigate the non-extremal black holes. The potential of the non-extremal black hole takes a barrier-well type between the inner and outer horizons, while it takes a simple barrier outside the outer horizon.

In this case the roots of  $V = 0$  are  $r = r_-, r_b, r_+$ , in sequence. Here the inner and outer horizons ( $r = r_-, r_+$ ) come from  $f = 0$ , and while  $r = r_b$  from  $\sqrt{2}f' - 2(1 - f) = 0$ . As is shown in Fig. 5,  $V(r, \epsilon, Q)$  for  $Q^2 = Q_e^2/2$  increases without bound in height as  $\epsilon$  approaches to 2 (the upper limit). The graphical analysis in Fig. 6 and 7 shows that  $V(r, \epsilon, Q)$  for  $Q^2 = Q_e^2/2$  are shifted to  $V$ -axis as  $\epsilon$  decreases. However, the shape of a barrier-well is not significantly changed under the variation of  $\epsilon$ .

In this section we study mainly the inner structure of the  $\epsilon < 2$  charged black holes. It was shown that for  $\epsilon = 0$  the inner horizon is unstable, whereas the outer one is stable [9]. First we consider the region inside the black hole ( $r_- < r < r_+$ ). Fig. 8 shows that the inner structure of the non-extremal black hole has the nontrivial causal structure of the spacetime. It is very important to note that inside the black hole the radial coordinate ( $r$  or  $r^*$ ) is timelike, whereas the time ( $t$ ) is spacelike. Hence to quest the internal structure of black hole is an evolutionary problem. Two observers are shown falling through  $r = r_+$  into the interior region and then through the Cauchy horizon at  $r = r_-$ . FFO1 (FFO2) crosses the left (right) branch of  $r = r_-$ . An incident wave is scattered from the potential ( $V^{OUT}$ ) outside the outer horizon. The scattered waves by  $V^{OUT}$  will be rescattered into the hole, to give a tail with a power-law (in time) decay. Explicitly the power-law tails come from rescattering of the scattered waves off the weak potential far from the black hole. Since this decay rate is sufficiently slow, it plays a role to develop the infinite energy densities on the right branch [11]. On the other hand, the transmitted wave proceeds into the interior region where further scattering by  $V^{IN}$  occurs. This scattered (right-moving) wave is then rescattered at  $r^* \simeq 0$  to give a left-moving wave traveling near the  $v = \infty$  horizon. Here we investigate the way how the transmitted waves develop the infinite energy density near the  $u = \infty$  branch. In the non-extremal case ( $Q^2 < Q_e^2$ ), the tortoise coordinate is differently given by

$$r^* = \int^r \frac{dr}{f(r, \epsilon, Q)} = -\frac{1}{2\sqrt{2}} \int^y \frac{dy}{yf(y, \epsilon, Q)}, \quad (48)$$

where  $f(y, \epsilon, Q) = 1 - y + \tilde{A}y^{1+B}$  with  $\tilde{A} = \frac{Q^2}{4(2-\epsilon)}$ . Near both horizons,  $f(y, \epsilon, Q) \simeq f'(y_{\pm})(y - y_{\pm})$ . Substituting this into (48), one finds

$$y - y_{\pm} \simeq \pm \exp(\pm \kappa_{\pm} r^*), \quad (49)$$

where  $\kappa_{\pm}(r_{\pm}, \epsilon, Q) \equiv \pm f'(r_{\pm}) (= \mp 2\sqrt{2}f'(y_{\pm})y_{\pm})$  are the surface gravities at  $r = r_{\pm}$ . It is obvious from the graphical analysis in Fig.1 and Fig. 9 that  $\kappa_{\pm} > 0$ . Near the outer horizon the potential decreases exponentially as

$$V_{r \rightarrow r_+}^{IN}(r^*, \epsilon, Q) \simeq -2(2f'(y_+)y_+ + 1)f'(y_+) \exp(\kappa_+ r^*), \quad r^* \rightarrow -\infty (r \rightarrow r_+) \quad (50)$$

and near the inner horizon it takes the form

$$V_{r \rightarrow r_-}^{IN}(r^*, \epsilon, Q) \simeq 2(2f'(y_-)y_- + 1)f'(y_-) \exp(-\kappa_- r^*), \quad r^* \rightarrow \infty (r \rightarrow r_-). \quad (51)$$

We note that for the inner scattering analysis, the inner region ( $r_- < r < r_+$ ) should be changed into  $\infty > r^* > -\infty$ . It is useful to introduce the null coordinates ( $v = r^* + t, u = r^* - t$ ) to describe the inner structure of the charged black hole. The metric is then given by  $ds^2 = f dv du$ . As is shown in Fig.8, the Cauchy horizon  $r = r_-$  consists of two branches (the right with  $v = \infty$  and the left with  $u = \infty$ ). In order to find the energy density of the tachyon measured by a freely falling observer (FFO) with two-velocity  $U^\mu (U^\mu U_\mu = -1)$ , we have to consider the boundary conditions. Initially the tachyonic mode with  $\omega$  falls into the hole from the exterior region. A general perturbation is a superposition of these  $\omega$  modes. Considering  $t_\omega^{IN}(r^*, t) \sim t_\omega^{IN}(r^*) e^{-i\omega t}$ , one finds the equation for a particular frequency ( $\omega$ ) near the horizons

$$\left\{ \frac{\partial^2}{\partial r^{*2}} + \omega^2 \right\} t_\omega^{IN}(r^*) = 0. \quad (52)$$

Here we obtain purely the ingoing wave near the event horizon ( $r_+$ )

$$t_\omega^{IN}(r^*, t) |_{r_+} \equiv t_\omega^{IN}(u, v) |_{r_+} = T^{IN}(\omega) e^{-i\omega v}. \quad (53)$$

On the other hand, the boundary condition near the Cauchy horizon is

$$t_\omega^{IN}(r^*, t) |_{r_-} \equiv t_\omega^{IN}(u, v) |_{r_-} = e^{-i\omega v} + R^{IN}(\omega) e^{i\omega u}, \quad (54)$$

where the first term refers the ingoing mode into the left branch with  $u = \infty$ , while the second denotes the backscattered mode into the right branch with  $v = \infty$ . Here  $T^{IN}(\omega)$  and  $R^{IN}(\omega)$  are the transmission and reflection amplitudes for the inner scatterings. These altered boundary conditions arise from the fact that by virtue of the light-cone structure of the inner region, one can have only ingoing modes (by crossing the outer horizon) and none leaving it (by crossing the outer one in the reverse direction). We take into account

the general perturbation ( $\hat{t}^{IN}(r^*, t) = e^{-\sqrt{2}r} t^{IN}(r^*, t)$  in (14)) to obtain the energy density. However, one has  $e^{-\sqrt{2}r} \sim e^{-\sqrt{2}r_-}$  near the Cauchy horizon and thus can neglect it. This is given by the Fourier integral transform over the frequency  $\omega$

$$\hat{t}^{IN}(r^*, t) \sim t^{IN}(r^*, t) = \int t_{\omega}^{IN}(r^*) e^{-i\omega t} a(\omega) d\omega \quad (55)$$

with the mode amplitude  $a(\omega)$ . Considering the boundary condition (54) near the Cauchy horizon, this takes the form

$$t^{IN}(r^*, t) |_{r_-} = [t^r(v) + t^l(u)], \quad (56)$$

where  $t^r$  is a function of  $v(=r^*+t)$  and  $t^l$  is a function of  $u(=r^*-t)$ . The energy density measured by a FFO is dominated by [9]

$$\rho = T_{\alpha\beta}^t U^{\alpha} U^{\beta} \sim |U^{\alpha} t^{IN}_{,\alpha}|^2 \sim |U^{\alpha} t^{l,r}_{,\alpha}|^2. \quad (57)$$

When a FFO1 crosses the left branch ( $u \rightarrow \infty$ ) of the Cauchy horizon, one has

$$U^{\alpha} t^{l}_{,\alpha} \propto t^{l'}(u) \exp\left(\frac{\kappa_- u}{2}\right), \quad (58)$$

where the prime means the differentiation with respect to the given argument. In order to calculate  $t^{l'}(u)$ , we consider the deviation from the wave ( $t_{\omega}^{IN}(r^*) = \exp(-i\omega r^*)$ ) treating  $V_{r \rightarrow r_-}^{IN} \sim \exp(-\kappa_- r^*)$  in (51) as the infinitesimal perturbation. In this case one has a first-order scattering equation between two horizons :  $(d^2/dr^{*2} + \omega^2)t_{\omega}^{IN} = V_{r \rightarrow r_-}^{IN} t_{\omega}^{IN}$ . It is a standard method to use Green's function in solving the above equation. Following Ref.[12], we find  $t^{l'}(u) \propto \exp(-\frac{\kappa_- u}{2})$  as  $u \rightarrow \infty$ . Therefore this wave gives a finite energy density at the left Cauchy horizon. On the other hand, the energy density measured by a FFO2 who crosses the right ( $v \rightarrow \infty$ ) horizon is proportional to the square of

$$U^{\alpha} t^{r}_{,\alpha} \propto t^{r'}(v) \exp\left(\frac{\kappa_+ v}{2}\right). \quad (59)$$

In order to calculate  $t^{r'}(v)$ , one also consider the deviation from the wave ( $t_{\omega}^{IN}(r^*) = \exp(i\omega r^*)$ ) treating  $V_{r \rightarrow r_+}^{IN} \sim \exp(\kappa_+ r^*)$  in (50) as the infinitesimal perturbation. It is calculated as  $t^{r'}(v) \propto \exp(-\frac{\kappa_+ v}{2})$ . Substituting this into (59) leads to

$$U^{\alpha}t^r{}_{,\alpha} \propto \exp\left\{\frac{(\kappa_- - \kappa_+)v}{2}\right\}. \quad (60)$$

From the graphical analysis in Fig.1 and Fig.9, it is easily shown that  $-f'(r_-) > f'(r_+)$  ( $\kappa_- > \kappa_+$ ) for all  $\epsilon < 2, Q^2 < Q_e^2$ . This means that the slope of the metric function  $f$  at  $r = r_-$  is always greater than the slope of  $f$  at  $r = r_+$ . Thus (60) leads to a divergent energy density on the right Cauchy horizon. This shows that the monochromatic tachyon waves with small amplitude and purely ingoing near the event horizon in (53) develop the infinite energy density near Cauchy horizon. The FFO2 meets a divergent energy density when he (or she) crosses the right branch of inner horizon. This corresponds to the blueshift of tachyon. Further this means that the Cauchy horizon of the 2D charged black hole is unstable to the physical perturbations.

For the stability of outer horizon, one notes the shape of the potentials outside the the outer horizon ( $V^{OUT}$ ) in Fig. 5 to Fig. 7. Outside the horizon ( $r_+$ ), all of the potentials take the positive barriers. This means that the corresponding Schrödinger-type equations allow only the scattering states. Since one cannot find any exponentially growing mode, the outer horizon of non-extremal black hole is stable.

## VI. HAWKING TEMPERATURE

The Hawking temperature of a static black hole can be calculated in several ways [13]. Suppose that the metric takes the form

$$ds^2 = -\lambda dt^2 + \frac{dr^2}{f}. \quad (61)$$

Near the outer horizon  $r = r_+$ , one has  $\lambda \simeq \lambda'(r_+)(r - r_+)$  and  $f \simeq f'(r_+)(r - r_+)$ . We now set  $\tau = it$  and  $\rho = 2\sqrt{(r - r_+)/f'(r_+)}$ . The resulting metric is

$$ds^2 = d\rho^2 + \frac{\lambda'(r_+)f'(r_+)}{4}\rho^2 d\tau^2. \quad (62)$$

In order to avoid a conical singularity at  $\rho = 0$  we must identify  $\tau$  with period  $4\pi/\sqrt{\lambda'(r_+)f'(r_+)}$ . Thus the Hawking temperature is given by

$$T_H = \frac{\sqrt{\lambda'(r_+)f'(r_+)}}{4\pi}. \quad (63)$$

In our case,  $\lambda = f$ . Therefore the Hawking temperature takes the form

$$T_H = \frac{|f'(r_+)|}{4\pi}, \quad (64)$$

which leads to

$$T_H = \frac{e^{-2\sqrt{2}r_+}}{\sqrt{2}\pi} \left| \frac{M}{\sqrt{2}} - \frac{Q^2}{8} \left( \frac{4-\epsilon}{2-\epsilon} \right) e^{-(2-\epsilon)\sqrt{2}r_+} \right|. \quad (65)$$

As is expected, one finds that  $T_H^e \rightarrow 0$  in the extremal limit of  $Q^2 \rightarrow Q_e^2$  and  $r_+ \rightarrow r_o$ . In the case of  $\epsilon = 0$ , one finds the explicit form

$$T_H^Q = \frac{\sqrt{2}}{\pi} \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})}. \quad (66)$$

Further for the dilaton black hole ( $\epsilon = Q = 0$ ), the Hawking temperature corresponds to the statistical temperature

$$T_H^{Q \rightarrow 0, \epsilon \rightarrow 0} = \frac{1}{\sqrt{2}\pi}. \quad (67)$$

## VII. DISCUSSIONS

The 2D,  $\epsilon < 2$  extremal black holes have all zero Hawking temperature ( $T_H^e = 0$ ). However one finds the instability for 2D,  $\epsilon < 2$  extremal black hole, which originates from the barrier-well potentials. These potentials persist when the nonextremal black hole approaches the extremal limit. As is discussed in Ref.[14], the quantum stress tensor of a scalar field (instead of the tachyon) in the  $\epsilon = 0$ , 2D extremal black hole diverges at the horizon. This means that the  $\epsilon = 0$ , 2D extremal black hole is also quantum-mechanically unstable. This divergence can be better understood by the regarding an extremal black hole as the limit of a nonextremal one. A non-extremal black hole has an outer (event) and an inner (Cauchy) horizon, and these come together in the extremal limit. In this case, it finds that if one adjusts the quantum state of the scalar field so that the stress tensor is finite at the outer



horizon, it always diverges at the inner horizon. Thus it is not so surprising that in the extremal limit (when the two horizons come together) the divergence persists, although it has a softened form. By the similar way, it would be conjectured that the classical instability of  $\epsilon < 2$  extremal black holes originates from the instability (blueshift) of the inner horizon in the  $\epsilon < 2$  non-extremal black holes. The potential of the non-extremal black hole takes a barrier-well between the inner and outer horizons, while it takes a simple barrier outside the outer horizon. It is confirmed that the inner horizon is unstable, whereas the outer one is stable. When these coalesce, a barrier-well type potential appears outside the event horizon ( $r > r_o$ ). This induces the instability of the extremal black holes. Contrary to the 2D calculations, the 4D extremal black holes including Reissner-Nordström one are all classically stable [3]. This is so because all potentials are positive definite outside the horizons. Further, the quantum stress-energy tensor of a scalar field is recently calculated in the  $a = 0$  extremal Reissner-Nordström black hole [15]. The stress-energy appears to be regular on the horizon. This establishes that the  $a = 0$ , 4D extremal black hole is classically and quantum-mechanically stable.

In conclusion, although the 2D extremal charged black holes all have zero Hawking temperature, they cannot be the toy models for the stable endpoint of the Hawking evaporation.

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## FIGURES

Fig. 1: Three graphs of the metric function  $f(r, \epsilon, Q)$  for  $\epsilon = 0.5$ :  $Q^2 > Q_e^2$  (dashed line : ---, no root);  $Q^2 = Q_e^2$  (dotted line :- - - -, extremal case with a multiple root );  $Q^2 < Q_e^2$  (solid line : —, non-extremal case with two roots).

Fig. 2: Three graphs of extremal potentials ( $V_e(r, \epsilon)$ ) for  $\epsilon = 1.9$  (dashed line : ---),  $0.5$  (dotted line :- - - -), and  $-3$  (solid line : —). The potentials are zero at  $r_o(\epsilon) = -1.076, -0.299$ , and  $-0.119$  and they are all barrier-well types outside  $r_o(\epsilon)$ .

Fig. 3: The approximate potential ( $V_{app}^e(r^*, \epsilon)$ ). The asymptotically flat region is at  $r^* = \infty$ . This also takes a barrier-well type. This is localized at  $r^* = 0$ , falls to zero exponentially as  $r^* \rightarrow \infty$  and inverse-squarely as  $r^* \rightarrow -\infty$  (solid lines). The dotted line is used to connect two boundaries.

Fig. 4: Three  $\epsilon = 0$  graphs of potential for  $Q = 0.1$  (dashed line : ---),  $1$  (dotted line :- - - -), and  $\sqrt{2}$  (solid line : —). The corresponding event horizons are located at  $r_+ = -0.004, -0.056$ , and  $-0.245$  respectively. When  $M(=\sqrt{2}) > Q(Q = 0.1, 1)$ , the potentials outside the event horizon are simple barriers. However a barrier-well ( $V_e(r, \epsilon)$ ) appears as the nonextremal black hole (a simple barrier) approaches the extremal one ( $M = Q$ ).

Fig. 5:  $1 \leq \epsilon < 2$  graphs of the effective potential ( $V(r, \epsilon, Q)$ ) for  $Q^2 = Q_e^2/2$ :  $\epsilon = 1.9$  (dashed line : ---),  $\epsilon = 1.6$  (dotted line :- - - -), and  $\epsilon = 1.0$  (solid line : —). The potentials increase without bound in height as  $\epsilon$  approaches its upper limit ( $=2$ ).

Fig. 6:  $0.1 \leq \epsilon < 1$  graphs of the effective potential for  $Q^2 = Q_e^2/2$ :  $\epsilon = 0.8$  (dashed line : ---),  $\epsilon = 0.5$  (dotted line :- - - -), and  $\epsilon = 0.1$  (solid line : —). These take the barrier-well type ( $V^{IN}$ ) between two horizons, while it takes a simple potential barrier ( $V^{OUT}$ ) outside the outer horizon.

Fig. 7:  $\epsilon < 0$  graphs of the effective potential for  $Q^2 = Q_e^2/2$  :  $\epsilon = -0.5$  (dashed line : ---),  $\epsilon = -3$  (dotted line :- - -), and  $\epsilon = -10$  (solid line : —). These take also the barrier-well type ( $V^{IN}$ ) between two horizons, while it takes a simple potential barrier ( $V^{OUT}$ ) outside the outer horizon. As  $\epsilon$  decreases, the inner potentials are shifted to  $V$ -axis. And the shapes of outer potentials are not significantly changed. .

Fig. 8: Conformal diagram of a portion of the 2D non-extremal charged black hole space-time. Two observers are shown falling through  $r = r_+$  into the interior region and then through the Cauchy horizon at  $r = r_-$ . FFO1 (FFO2) crosses the left (right) branch of  $r = r_-$ . An incident wave is scattered from the potential ( $V^{OUT}$ ). The scattered wave by  $V^{OUT}$  will be rescattered into hole, to give a tail with a power-law (in time) decay. This decay in time is sufficiently slow so that infinite energy densities are developed on the right branch [11]. On the other hand, the transmitted wave proceeds into the interior region where further scattering by  $V^{IN}$  occurs. This right-moving wave is rescattered at  $r^* \simeq 0$  to give a left-moving wave traveling near the  $v = \infty$  horizon. Here we show the way that these waves develop the infinite energy density (measured by FFO2) near the  $u = \infty$  branch.

Fig. 9: Three graphs of non-extremal  $f(r, \epsilon, Q)$  for  $Q^2 = Q_e^2/2$  :  $\epsilon = 1.5$  (dashed line : ---), 0.1 (dotted line :- - -), and  $-3$  (solid line : —). It is obvious that  $-f'(r_-) > f'(r_+)$  for all  $\epsilon < 2$ . As  $\epsilon$  approaches 2, the decreasing rate of  $f$  at the inner horizon ( $r = r_-$ ) is always greater than the increasing rate of  $f$  at the outer horizon ( $r = r_+$ ).